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SERIES DE FOURIER

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TRANSFORMADA DE FOURIER DE LAS DERIVADA PARCIALES

Consideremos $u:\mathbb{R}^2 \to \mathbb{R}$, talque: $u = u(x,t)$ una función definida en $0 \le x \le L$, y $t \ge 0$

Con respecto al intervalo , podemos reflejar par o impar por consiguiente se defina la transformación de Fourier de cosenos y senos respectivamente.

Luego, existe la transformada de las derivadas parciales.

$\dfrac{du}{dx}, \dfrac{d^2u}{dx^2}, \dfrac{d^3u}{dx^3}, \cdots$ definida en $0 \le x \le L$ (llamada transformada de Fourier finita)

i) $\boxed{\mathscr{F}_S\left\{\dfrac{du}{dx}\right\} = -\lambda\, \mathscr{F}_C\{\, u \,\}}$

En efecto, como: $\mathscr{F}_S\left\{\dfrac{du}{dx}\right\} = \int_0^L \dfrac{du}{dx}\,\mathrm{sen}(\lambda x)\,dx$

Por el método por partes: $\mu = \mathrm{sen}(\lambda x)$ $\quad dv = \dfrac{du}{dx}(x,t)\,dx$

$d\mu = \lambda\cos(\lambda x)\,dx \quad v = u(x,t)$

$$\Rightarrow \mathscr{F}_S\left\{\frac{du}{dx}\right\} = u(x,t)\,\mathrm{sen}(\lambda x)\Big|_0^L - \int_0^L \lambda\, u\cos(\lambda x)\,dx \qquad \lambda = \frac{n\pi}{L}, \quad \forall\, n = 1,2,3,\cdots$$

$$= -\lambda\int_0^L u(x,t)\cos(\lambda x)\,dx = -\lambda\,\mathscr{F}_C\{\, u \,\}$$

ii) $\boxed{\mathscr{F}_C\left\{\dfrac{du}{dx}\right\} = \lambda\,\mathscr{F}_S\{\, u \,\} - [u(0,t) - u(L,t)\cos(n\pi)]}$

En efecto,

$\mathscr{F}_C\left\{\dfrac{du}{dx}\right\} = \int_0^L \dfrac{du}{dx}\cos(\lambda x)\,dx$, por el método por partes: $\mu = \cos(\lambda x)$ $\quad dv = \dfrac{du}{dx}dx$

$d\mu = -\lambda\,\mathrm{sen}(\lambda x)\,dx \quad v = u$

$$\Rightarrow \mathscr{F}_C\left\{\frac{du}{dx}\right\} = u(x,t)\cos(\lambda x)\Big|_0^L - \int_0^L (-\lambda)\,u(x,t)\,\mathrm{sen}(\lambda x)\,dx$$

$$= u(L,t)\cos(\lambda L) - u(0,t)\cos(\lambda(0)) + \lambda\int_0^L u(x,t)\,\mathrm{sen}(\lambda x)\,dx$$

$$= \lambda\mathscr{F}_S\{\, u \,\} - [u(0,t) - u(L,t)\cos(n\pi)] \quad , \quad \text{pues } \lambda = \frac{n\pi}{L} \quad \forall\, n = 1,2,3,\cdots$$

iii) $\boxed{\mathscr{F}_S\left\{\dfrac{d^2u}{dx^2}\right\} = -\lambda^2\ \mathscr{F}_S\{\,u\,\} + \lambda\left[\,u(0,t)-u(L,t)\cos(n\pi)\,\right]}$

$$\mathscr{F}_S\left\{\frac{d^2u}{dx^2}\right\} = \mathscr{F}_S\left\{\frac{d}{d\,x}\left(\frac{d\,u}{d\,x}\right)\right\} = -\lambda\ \mathscr{F}_C\left\{\frac{d\,u}{d\,x}\right\} \qquad \text{, por } i)$$

$$= -\lambda\left[\lambda\,\mathscr{F}_S\{\,u\,\} - u(0,t) + u(L,t)\cos(n\,\pi)\right] \qquad \text{, por } ii)$$

$$= -\lambda^2\,\mathscr{F}_S\{\,u\,\} + \lambda\left[u(0,t)-u(L,t)\cos(n\,\pi)\right]$$

iv) $\boxed{\mathscr{F}_C\left\{\dfrac{d^2u}{dx^2}\right\} = -\lambda^2\ \mathscr{F}_C\{\,u\,\} - \left[\,u_x(0,t)-u_x(L,t)\cos(n\pi)\,\right]}$

$$\mathscr{F}_C\left\{\frac{d^2u}{dx^2}\right\} = \mathscr{F}_C\left\{\frac{d}{d\,x}\left(\frac{d\,u}{d\,x}\right)\right\} = \lambda\ \mathscr{F}_S\left\{\frac{d\,u}{d\,x}\right\} - \left[\frac{d\,u(0,t)}{d\,x} - \frac{d\,u(L,t)}{d\,x}\cos(n\pi)\right]$$

$$= \lambda\left(-\lambda\ \mathscr{F}_C\{\,u\,\}\right) - \left[\frac{d\,u(0,t)}{d\,x} - \frac{d\,u(L,t)}{d\,x}\cos(n\pi)\right] \qquad \text{, por } i)$$

$$= -\lambda^2\ \mathscr{F}_C\{\,u\,\} - \left[\,u_x(0,t)-u_x(L,t)\cos(n\pi)\,\right]$$

Para las

$\dfrac{du}{dt}, \dfrac{d^2u}{dt^2}, \dfrac{d^3u}{dt^3}, \cdots$ definida en $0 \le x \le L$

i) $\boxed{\mathscr{F}_S\left\{\dfrac{d\,u}{d\,t}\right\} = \dfrac{d}{dt}\,\mathscr{F}_S\{\,u\,\}}$

En efecto: $\mathscr{F}_S\left\{\dfrac{du}{dt}\right\} = \displaystyle\int_0^L \dfrac{du}{dt}\,\text{sen}(\lambda x)\,dx = \dfrac{d}{dt}\int_0^L u\,\text{sen}(\lambda x)\,dx = \dfrac{d}{dt}\,\mathscr{F}_S\{\,u\,\}$

De la misma forma se prueban:

ii) $\boxed{\mathscr{F}_C\left\{\dfrac{d\,u}{d\,t}\right\} = \dfrac{d}{dt}\,\mathscr{F}_C\{\,u\,\}}$

iii) $\boxed{\mathscr{F}_S\left\{\dfrac{d^2u}{d\,t^2}\right\} = \dfrac{d^2}{d\,t^2}\,\mathscr{F}_S\{\,u\,\}}$

iv) $\boxed{\mathscr{F}_C\left\{\dfrac{d^2u}{d\,t^2}\right\} = \dfrac{d^2}{d\,t^2}\,\mathscr{F}_C\{\,u\,\}}$

Para el caso aplicativo, las funciones f definidas en el intervalo finito $0 \le t \le L$

Es decir: $f: I \subset \mathbb{R} \to \mathbb{R}$,donde: $I = [0, L]$

Estas funciones tienen transformadas de Fourier finita, entonces es posible efectuar reflexiones par e impar y por ende se define la transformada de Fourier finitas de senos y cosenos de sus derivadas $f\,'(t)$, $f\,''(t)$, $f\,'''(t)$, $\cdots$ similarmente, como en el caso anterior:

i) $\boxed{\mathscr{F}_S\{\,f\,'(t)\,\} = -\lambda\ \mathscr{F}_C\{\,f(t)\,\}}$

En efecto, como: $\mathscr{F}_S\{\,f\,'(t)\,\} = \displaystyle\int_0^L f\,'(t)\,\text{sen}(\lambda t)\,dt$

Por el método por partes: $\mu = \text{sen}(\lambda t)$ $\quad d\nu = f\,'(t)\,dt$

$d\mu = \lambda\cos(\lambda t)\,dt$ $\quad \nu = f(t)$

$$\Rightarrow \mathscr{F}_S\{\,f\,'(t)\,\} = f(t)\,\text{sen}(\lambda t)\Big|_0^L - \int_0^L \lambda\ f(t)\cos(\lambda x)\,dx \qquad \lambda = \frac{n\,\pi}{L}, \qquad \forall\, n = 1,2,3,\cdots$$

$$= -\lambda\int_0^L f(t)\cos(\lambda\,t)\,dt = -\lambda\ \mathscr{F}_C\{\,f(t)\,\}$$

ii) $\boxed{\mathscr{F}_C\{\,f\,'(t)\,\} = \lambda\ \mathscr{F}_S\{\,f(t)\,\} - \left[\,f(0) - f(L)\cos(n\pi)\right]}$

En efecto,

$\mathscr{F}_C\{\,f\,'(t)\,\} = \displaystyle\int_0^L f\,'(t)\cos(\lambda t)\,dt$, por el método por partes: $\mu = \cos(\lambda t)$ $\quad d\nu = f\,'(t)dt$

$d\mu = -\lambda\,\text{sen}(\lambda t)\,dt$ $\quad \nu = f(t)$

$$\Rightarrow \mathscr{F}_C\{\,f\,'(t)\,\} = f(t)\cos(\lambda t)\Big|_0^L - \int_0^L (-\lambda)\ f(t)\,\text{sen}(\lambda t)\,d\,t$$

$$= f(L)\cos(\lambda L) - f(0)\cos(\lambda(0)) + \lambda\int_0^L f(t)\,\text{sen}(\lambda x)\,dx$$

$$= \lambda\,\mathscr{F}_S\{\,u\,\} - \left[\,f(0) - f(L)\cos(n\,\pi)\right] \quad , \quad \text{pues} \quad \lambda = \frac{n\,\pi}{L} \quad \forall\, n = 1,2,3,\cdots$$

iii) $\boxed{\mathscr{F}_S\{\,f\,''(t)\,\} = -\lambda^2\ \mathscr{F}_S\{\,f(t)\,\} + \lambda\left[\,f(0) - f(L)\cos(n\pi)\,\right]}$

$$\mathscr{F}_S\{\,f\,''(t)\,\} = \mathscr{F}_S\left\{\frac{d}{d\,t}\,f\,'(t)\right\} = -\lambda\ \mathscr{F}_C\{\ f\,'(t)\,\} \qquad \text{, por } i)$$

$$= -\lambda\left[\lambda\,\mathscr{F}_S\{\,f(t)\,\} - u(0,t) + u(L,t)\cos(n\,\pi)\right] \qquad \text{, por } ii)$$

$$= -\lambda^2\,\mathscr{F}_S\{\,f(t)\,\} + \lambda\left[u(0,t) - u(L,t)\cos(n\,\pi)\right]$$

iv) $\boxed{\mathscr{F}_C\{ f''(t)\} = -\lambda^2\ \mathscr{F}_C\{ f(t)\} - [\ f'(0) - f'(L)\cos(n\pi)\ ]}$

$$\mathscr{F}_C\{ f''(t)\} = \mathscr{F}_C\left\{ \frac{d}{dt} f'(t) \right\} = \lambda\ \mathscr{F}_S\{ f'(t)\} - [\ f'(0) - f'(L)\cos(n\pi)]$$

$$= \lambda\left(-\lambda\ \mathscr{F}_C\{ f(t)\ \}\right) - [\ f'(0) - f'(L)\cos(n\pi)]\ ,\ \text{por } i)$$

$$= -\lambda^2\ \mathscr{F}_C\{ f(t)\} - [\ f'(0) - f'(L)\cos(n\pi)\ ]$$

EJERCICIOS RESUELTOS

1. Calcular la transformada de Fourier

$$\mathscr{F}\{ e^{-a|t|}\} = \frac{2a}{\lambda^2 + a^2} \quad \text{, es decir} \quad \mathscr{F}^{-1}\left\{ \frac{1}{\lambda^2 + a^2} \right\} = \frac{e^{-a|t|}}{2a}$$

Solución:

$$\mathscr{F}\{ e^{-a|t|}\} = \int_{-\infty}^{\infty} e^{-a|t|}\ e^{-i\lambda t}\ dt = \int_{-\infty}^{0} e^{-a(-t)}\ e^{-i\lambda t}\ dt + \int_{0}^{\infty} e^{-at}\ e^{-i\lambda t}\ dt$$

$$= \int_{-\infty}^{0} e^{-(i\lambda - a)t}\ dt + \int_{0}^{\infty} e^{-(i\lambda + a)t}\ dt$$

$$= -\frac{1}{i\lambda - a} e^{-(i\lambda - a)t}\Big|_{-\infty}^{0} - \frac{1}{i\lambda + a} e^{-(i\lambda + a)t}\Big|_{0}^{\infty}$$

$$= -\frac{1}{i\lambda - a}[1-0] - \frac{1}{i\lambda + a}[0-1] = -\frac{1}{i\lambda - a} + \frac{1}{i\lambda + a} = -\frac{1}{i\lambda - a} + \frac{1}{i\lambda + a}$$

$$= \frac{-(i\lambda + a) + (i\lambda - a)}{(i\lambda - a)(i\lambda + a)} = \frac{-2a}{(i\lambda)^2 - a^2} = \frac{-2a}{-\lambda^2 - a^2} = \frac{2a}{\lambda^2 + a^2}$$

Entonces: $\quad \mathscr{F}\{ e^{-a|t|}\} = \dfrac{2a}{\lambda^2 + a^2}$

Además:

$$\mathscr{F}^{-1}\left\{ \frac{2a}{\lambda^2 + a^2} \right\} = e^{-a|t|} \Rightarrow 2a \cdot \mathscr{F}^{-1}\left\{ \frac{1}{\lambda^2 + a^2} \right\} = e^{-a|t|} \Rightarrow \mathscr{F}^{-1}\left\{ \frac{1}{\lambda^2 + a^2} \right\} = \frac{e^{-a|t|}}{2a}$$

2. Utilizando la trasformada de Fourier resolver $y' - 4y = f(t)$

donde $f(t) = \begin{cases} e^{-4t} & ,\text{si}\, t \geq 0 \\ 0 & ,\text{si}\, t < 0 \end{cases}$

Solución:

Es un dominio infinito, es una ecuación de primer orden.
El dominio de la Ecuación diferencial es de $(-\infty, \infty)$ es correspondiente la transformada de Fourier para $(-\infty, \infty)$.

Tomando la transformada de Fourier:

$\mathscr{F}\{ y' - 4y \} = \mathscr{F}\{ f(t) \}$, es un operador lineal

$$\Rightarrow \mathscr{F}\{ y' \} - 4\mathscr{F}\{ y \} = \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t} dt$$

$$\Rightarrow i\lambda \mathscr{F}\{ y \} - 4\mathscr{F}\{ y \} = \underbrace{\int_{-\infty}^{0} 0\, e^{-i\lambda t} dt}_{=0} + \int_{0}^{\infty} e^{-4t}\, e^{-i\lambda t} dt$$

$$\Rightarrow \mathscr{F}\{ y \}(i\lambda - 4) = \int_{0}^{\infty} e^{-(4+i\lambda)t}\, dt$$

$$\Rightarrow \mathscr{F}\{ y \}(i\lambda - 4) = -\frac{1}{4+i\lambda} e^{-(4+i\lambda)t} \Big|_{0}^{\infty}$$

$$\Rightarrow \mathscr{F}\{ y \}(i\lambda - 4) = -\frac{1}{4+i\lambda}\Big[\underbrace{e^{-(4+i\lambda)\infty}}_{=0} - \underbrace{e^{-(4+i\lambda)(0)}}_{=1} \Big]$$

$$\Rightarrow \mathscr{F}\{ y \}(i\lambda - 4) = \frac{1}{4+i\lambda}$$

$$\Rightarrow \mathscr{F}\{ y \} = \frac{1}{(i\lambda+4)(i\lambda-4)}$$

$$\Rightarrow \mathscr{F}\{ y \} = \frac{1}{(i\lambda)^2 - 4^2}$$

$$\Rightarrow \mathscr{F}\{ y \} = -\frac{1}{\lambda^2 + 16}$$

Aplicando la transformada inversa:

$$y = \mathscr{F}^{-1}\left\{ -\frac{1}{\lambda^2+16} \right\} = -\mathscr{F}^{-1}\left\{ \frac{1}{\lambda^2+4^2} \right\} = -\frac{e^{-4|t|}}{2(4)} = -\frac{1}{8} e^{-4|t|}$$

3. Resolver la siguiente Ecuación diferencial Parcial con las condiciones siguientes

$$\frac{d^2u}{dt^2} = a^2 \frac{d^2u}{dx^2} \quad , \quad 0 \leq x \leq L \;, \quad t \geq 0 .$$

a.c. $u(0,t) = 0$, $u(L,t) = 0$

$u(x,0) = f(x)$, $u_t(x,0) = g(x)$

$$\mathscr{F}_S\left\{ \frac{\partial^2 u(x,t)}{\partial x^2} \right\} = -\lambda^2 \mathscr{F}_S\{ u(x,t) \} - [u(0,t) - u(L,t)\cos(n\pi)]$$

$$\mathscr{F}_C\left\{ \frac{d^2u}{dt^2} \right\} = \frac{d^2u}{dt^2} \mathscr{F}_C\{ u(x,t) \}$$

$$\mathscr{F}_C\left\{ \frac{d^2u}{dx^2} \right\} = -\lambda^2\, \mathscr{F}_C\{ u \} - [\, u_x(0,t) - u_x(L,t)\cos(n\pi) \,]$$

Se tiene $u(0,t) = 0$, $u(L,t) = 0$

Si es derivadas parciales las condiciones utilizamos la $\mathscr{F}_C$, entonces utilizamos la $\mathscr{F}_S$

$$\mathscr{F}_S\left\{ \frac{d^2u}{dt^2} \right\} = \frac{d^2u}{dt^2} \mathscr{F}_S\left\{ a^2 \frac{d^2u}{dx^2} \right\} \quad \Rightarrow \frac{d^2}{dt^2} \mathscr{F}_S\{ u \} = a^2 \mathscr{F}_S\left\{ \frac{d^2u}{dx^2} \right\}$$

$$\Rightarrow \frac{d^2}{dt^2} \mathscr{F}_S\{ u \} = a^2 \left[-\lambda^2\, \mathscr{F}_S\{ u \} + \lambda [\, u(0,t) - u(L,t)\cos(n\pi) \,] \right]$$

Como: $u(0,t) = 0$, $u(L,t) = 0$

$$\Rightarrow \frac{d^2}{dt^2} \mathscr{F}_S\{ u \} = -a^2\lambda^2\, \mathscr{F}_S\{ u \} \Rightarrow \frac{d^2}{dt^2} \mathscr{F}_S\{ u \} + a^2\lambda^2\, \mathscr{F}_S\{ u \} = 0$$

Es una ecuación línea homogénea de 2do orden, por el polinomio característico, será:

$$r^2 + a^2\lambda^2 = 0 \qquad \Rightarrow \qquad \boxed{r = \pm\, a\, i\, \lambda}$$

Luego:

$$\boxed{\mathscr{F}_S\{ u \} = C_1 \cos(a\lambda t) + C_2 \mathrm{sen}(a\lambda t)} \quad \cdots(I)$$

Donde; $\lambda = \dfrac{n\pi}{p} = \dfrac{n\pi}{L} \qquad \Delta\lambda = \dfrac{\pi}{L}$

Calculemos C_1 y C_2.

Como: $\mathcal{F}_S\{\, u(x,0)\,\} = \mathcal{F}_S\{\, f(x)\,\}$

$$\mathcal{F}_S\{\, u(x,t)\,\} = C_1\cos(a\lambda t) + C_2\text{sen}(a\lambda t) \quad \Rightarrow \mathcal{F}_S\{\, f(x)\,\} = C_1\cos(0) + C_2\text{sen}(0) = C_1$$

Por lo tanto $\boxed{C_1 = \mathcal{F}_S\{\, f(x)\,\}}$

$$\frac{d}{dt}\mathcal{F}_S\{\, u(x,t)\,\} = \mathcal{F}_S\left\{\frac{d}{dt}u(x,t)\right\} = \mathcal{F}_S\{\, u_t(x,t)\,\}$$

$$\frac{d}{dt}\mathcal{F}_S\{\, u\,\} = -C_1\, a\,\lambda\,\text{sen}(a\lambda t) + a\,\lambda\, C_2\cos(a\lambda t)$$

$$\Rightarrow \mathcal{F}_S\{\, u_t(x,t)\,\} = -C_1\, a\,\lambda\,\text{sen}(a\lambda t) + a\,\lambda\, C_2\cos(a\lambda t)$$

$$\mathcal{F}_S\{\, u_t(x,0)\,\} = -C_1\, a\,\lambda\,\text{sen}(0) + a\,\lambda\, C_2\cos(0)$$ pues $u_t(x,0) = g(x)$

$$\mathcal{F}_S\{\, g(x)\,\} = a\,\lambda\, C_2\cos(0)$$

Luego: $\boxed{C_2 = \dfrac{1}{a\lambda}\mathcal{F}_S\{\, g(x)\,\}}$

En consecuencia en (I):

$$\mathcal{F}_S\{\, u(x,t)\,\} = \mathcal{F}_S\{\, f(x)\,\}\cdot\cos(a\lambda t) + \frac{1}{a\lambda}\mathcal{F}_S\{\, g(x)\,\}\cdot\text{sen}(a\lambda t)$$

Como se utiliza senos entonces utilizaremos la inversa de la transformada de senos.

$$u(x,t) = \frac{2}{\pi}\int_0^\infty F_S(\lambda)\,\text{sen}(\lambda x)\, d\lambda = \frac{2}{\pi}\int_0^\infty \mathcal{F}_S\{\, u(x,t)\,\}\,\text{sen}(\lambda x)\, d\lambda$$

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \left[\mathcal{F}_S\{\, f(x)\,\}\cdot\cos(a\lambda t) + \frac{1}{a\lambda}\mathcal{F}_S\{\, g(x)\,\}\cdot\text{sen}(a\lambda t)\right]\text{sen}(\lambda x)\, d\lambda$$

4. Resolver la siguiente Ecuación diferencial Parcial con las condiciones siguientes

$$\frac{du}{dt} = \frac{d^2u}{dx^2} \qquad , \qquad 0 \le x \le 3 \;\; , \quad t \ge 0\,.$$

a.c. $u(0,t) = u(3,t) = 0$,

$$u(x,0) = \begin{cases} x & ,\text{si } \; 0 < x < 1 \\ 0 & ,\text{si } \; 1 < x < 3 \end{cases},$$

Solución:
Es la transformada de Fourier de senos

$$\mathcal{F}_S\left\{\frac{du}{dt}\right\} = \mathcal{F}_S\left\{\frac{d^2u}{dx^2}\right\}$$

$$\frac{d}{dt}\mathcal{F}_S\{\, u\,\} = -\lambda^2\,\mathcal{F}_S\{\, u\,\} + \lambda\left[\, u(0,t) - u(3,t)\cos(n\pi)\,\right]$$

Como: $u(0,t) = u(3,t) = 0$

$$\Rightarrow \frac{d}{dt}\mathcal{F}_S\{\, u\,\} = -\lambda^2\,\mathcal{F}_S\{\, u\,\}$$

Es una Ecuación de 1er. Orden de variables separables ordinarias.

$$\frac{d\,\mathcal{F}_S\{\, u\,\}}{\mathcal{F}_S\{\, u\,\}} = -\lambda^2\, dt$$

Integrando:

$$\int\frac{d\,\mathcal{F}_S\{\, u\,\}}{\mathcal{F}_S\{\, u\,\}} = \int -\lambda^2\, dt \;\Rightarrow\; Ln\left(\mathcal{F}_S\{\, u\,\}\right) = -\lambda^2\, t + C \;\Rightarrow\; \mathcal{F}_S\{\, u\,\} = e^{-\lambda^2 t + C}$$

$$\Rightarrow \mathcal{F}_S\{\, u\,\} = e^{-\lambda^2 t}\cdot e^C \;\Rightarrow\; \mathcal{F}_S\{\, u\,\} = C'\cdot e^{-\lambda^2 t}$$

Calculemos C'.

$$\mathcal{F}_S\{\, u(x,0)\,\} = \int_0^1 x\,\text{sen}(\lambda\, x)\, dx + \int_1^3 (0)\cos(\lambda\, x)\, dx$$

$\mathcal{F}_S\{\, u(x,0)\,\} = \int_0^1 x\,\text{sen}(\lambda\, x)\, dx$ Por el métodos de partes $\mu = x$ $\quad dv = \text{sen}(\lambda x)\, dx$

$d\mu = dx \quad v = -\dfrac{1}{\lambda}\cos(\lambda x)$

$$\mathcal{F}_S\{\, u(x,0)\,\} = -\frac{1}{\lambda}\, x\cos(\lambda\, x)\Big|_0^1 + \frac{1}{\lambda}\int_0^1\cos(\lambda\, x)\, dx$$

$$\Rightarrow \mathscr{F}_S\{\, u(x,0)\,\} = -\frac{1}{\lambda}\cos(\lambda) + \frac{1}{\lambda^2}\operatorname{sen}(\lambda\, x)\Big|_0^1$$

$$\Rightarrow \mathscr{F}_S\{\, u(x,0)\,\} = -\frac{1}{\lambda}\cos(\lambda) + \frac{1}{\lambda^2}\operatorname{sen}(\lambda)$$

Como: $\mathscr{F}_S\{\, u(x,t)\,\} = C' \cdot e^{-\lambda^2 t} \Rightarrow \mathscr{F}_S\{\, u(x,0)\,\} = C' \cdot e^{-\lambda^2 (0)} \Rightarrow \mathscr{F}_S\{\, u(x,0)\,\} = C'$

$$\Rightarrow C' = -\frac{1}{\lambda}\cos(\lambda) + \frac{1}{\lambda^2}\operatorname{sen}(\lambda)$$

$$\Rightarrow C' = \frac{1}{\lambda^2}\left[\operatorname{sen}(\lambda) - \lambda\cos(\lambda)\right]$$

Entonces: $\mathscr{F}_S\{\, u\,\} = \frac{1}{\lambda^2}\left[\operatorname{sen}(\lambda) - \lambda\cos(\lambda)\right]\cdot e^{-\lambda^2 t}$

Luego, la integral de Fourier de senos y formula de inversión es:

$$u(x,t) = \frac{2}{\pi}\int_0^\infty \mathscr{F}_S\{\, u(x,t)\,\}\operatorname{sen}(\lambda x)\, d\lambda$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda^2}\left[\operatorname{sen}(\lambda) - \lambda\cos(\lambda)\right] e^{-\lambda^2 t}\operatorname{sen}(\lambda x)\, d\lambda$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{\frac{\pi^2 n^2}{L^2}}\left[\operatorname{sen}\left(\frac{n\pi}{L}\right) - \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}\right)\right] e^{-\left(\frac{n\pi}{L}\right)^2 t}\operatorname{sen}\left(\frac{n\pi}{L}x\right)\Delta\lambda$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{3^2}{\pi^2 n^2}\left[\operatorname{sen}\left(\frac{n\pi}{3}\right) - \frac{n\pi}{3}\cos\left(\frac{n\pi}{3}\right)\right] e^{-\frac{n^2\pi^2}{3^2}t}\operatorname{sen}\left(\frac{n\pi}{3}x\right)\cdot\frac{\pi}{3}$$

$$\Rightarrow u(x,t) = \frac{2(3)}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\operatorname{sen}\left(\frac{n\pi}{3}\right) - \frac{n\pi}{3}\cos\left(\frac{n\pi}{3}\right)\right] e^{-\frac{n^2\pi^2}{9}t}\operatorname{sen}\left(\frac{n\pi}{3}x\right)$$

EJERCICIOS

1. Resolver la siguiente Ecuación diferencial Parcial con las condiciones siguientes

$$\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2} \quad , \quad 0 \le x \le L \ , \quad t \ge 0 .$$

a.c. $u(0,t) = 0$, $u(L,t) = 0$

$u(x,0) = x$, $u_t(x,0) = \frac{x}{2}$

2. Resolver la siguiente Ecuación diferencial Parcial con las condiciones siguientes

$$\frac{du}{dt} = 2\frac{d^2u}{dx^2} \quad , \quad \text{a.c.} \quad 0 \le x \le 2 \ , \quad t \ge 0 .$$

$u(0,t) = u(2,t) = 0$,

$u(x,0) = 10\operatorname{sen}(4\pi x)$